The Big Picture

Chapter 3

Examining Computational Problems

- We want to examine a given computational problem and see how difficult it is.
- Then we need to compare problems
- Problems appear different

- We want to cast them into the same kind of problem
    - decision problems
    - in particular, language recognition problem

Decision Problems

A ***decision problem*** is simply a problem for which the answer is yes or no (True or False). A ***decision procedure*** answers a decision problem.

Examples:

- Given an integer *n*, does *n* have a pair of consecutive integers as factors?

- The language recognition problem: Given a language *L* and a string *w*, is *w* in *L*?

↑

Our focus

The Power of Encoding

- For problem already stated as decision problems.
  - encode the inputs as strings and then define a language that contains exactly the set of inputs for which the desired answer is yes.
- For other problems, must first reformulate the problem as a decision problem, then encode it as a language recognition task

Everything is a String

Pattern matching on the web:

- Problem: Given a search string *w* and a web document *d*, do they match? In other words, should a search engine, on input *w*, consider returning *d*?

- The language to be decided: {<*w*, *d*> : *d* is a candidate match for the query *w*}

Everything is a String

Does a program always halt?

- Problem: Given a program *p*, written in some some standard programming language, is *p* guaranteed to halt on all inputs?

- The language to be decided:

$$HP_{ALL} = \{p : p \text{ halts on all inputs}\}$$

Everything is a String

What If we're not working with strings?

Anything can be encoded as a string.

<*X*> is the string encoding of *X*.
<*X*, *Y*> is the string encoding of the pair *X*, *Y*.

Everything is a String

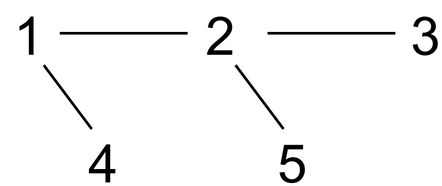
Primality Testing

- Problem: Given a nonnegative integer *n*, is it prime?
- An instance of the problem: Is 9 prime?
- To encode the problem we need a way to encode each instance: We encode each nonnegative integer as a binary string.
- The language to be decided:

  PRIMES = {*w* : *w* is the binary encoding of a prime number}.

Everything is a String

- Problem: Given an undirected graph *G*, is it connected?
- Instance of the problem:



- Encoding of the problem: Let *V* be a set of binary numbers, one for each vertex in *G*. Then we construct $\langle G \rangle$ as follows:
  - Write $|V|$ as a binary number,
  - Write a list of edges,
  - Separate all such binary numbers by "/".

101/1/10/10/11/1/100/10/101

- The language to be decided: CONNECTED = $\{w \in \{0, 1, /\}^* : w = n_1/n_2/\ldots n_i$, where each n_i is a binary string and *w* encodes a connected graph, as described above$\}$.

Turning Problems Into Decision Problems

Casting multiplication as decision:

- Problem: Given two nonnegative integers, compute their product.
- Reformulation: Transform computing into verification.
- The language to be decided:

  L = {w of the form:
  $<\mathit{integer}_1>\mathtt{x}<\mathit{integer}_2>=<\mathit{integer}_3>$, where:
  $\mathit{integer}_n$ is any well formed integer, and
  $\mathit{integer}_3 = \mathit{integer}_1 * \mathit{integer}_2$}

```
12x9=108
12=12
12x8=108
```

Turning Problems Into Decision Problems

Casting sorting as decision:

- Problem: Given a list of integers, sort it.
- Reformulation: Transform the sorting problem into one of examining a pair of lists.
- The language to be decided:

$L = \{w_1 \# w_2 : \exists n \geq 1$
$(w_1$ is of the form $<int_1, int_2, \ldots int_n>$,
w_2 is of the form $<int_1, int_2, \ldots int_n>$, and
w_2 contains the same objects as w_1 and
w_2 is sorted$)\}$

Examples:

```
1,5,3,9,6#1,3,5,6,9
1,5,3,9,6#1,2,3,4,5,6,7
```

Turning Problems Into Decision Problems

Casting database querying as decision:

- Problem: Given a database and a query, execute the query.
- Reformulation: Transform the query execution problem into evaluating a reply for correctness.
- The language to be decided:

L = {d # q # a:
d is an encoding of a database,
q is a string representing a query, and
a is the correct result of applying q to d}

Example:

```
(name, age, phone), (John, 23, 567-1234)
(Mary, 24, 234-9876)#(select name age=23)#
(John)
```

The Traditional Problems and their Language Formulations are Equivalent

By equivalent we mean that either problem can be ***reduced to*** the other.

If we have a machine to solve one, we can use it to build a machine to do the other using just the starting machine and other functions that can be built using a machine of equal or lesser power.

An Example

Consider the multiplication example:

$L = \{w$ of the form:

$<integer_1> \times <integer_2> = <integer_3>$, where:

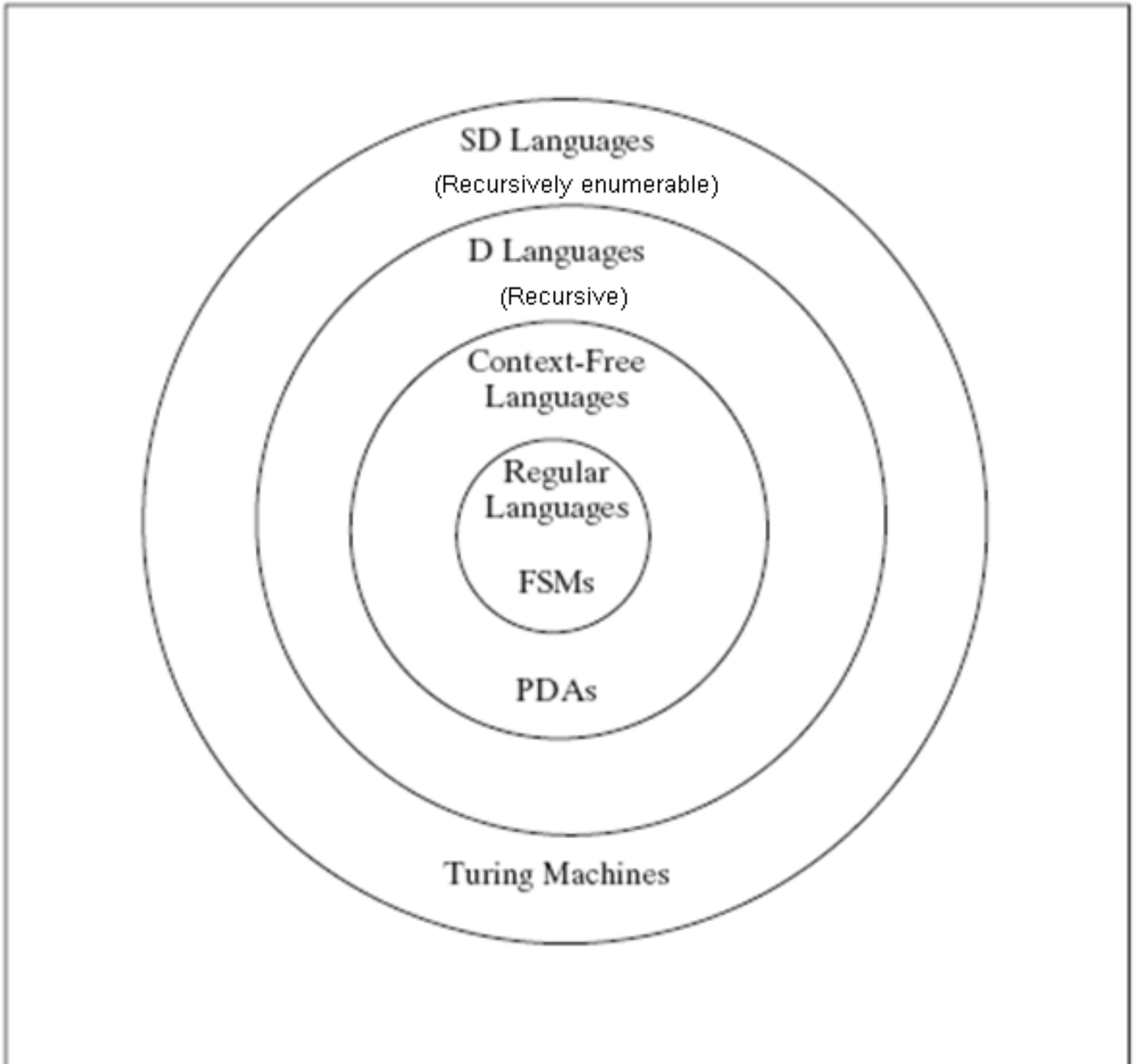
$integer_n$ is any well formed integer, and

$integer_3 = integer_1 * integer_2\}$

Given a multiplication machine, we can build the language recognition machine:

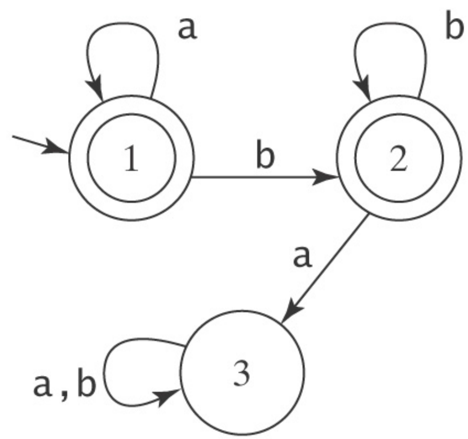
Given the language recognition machine, we can build a multiplication machine:

Languages and Machines



Finite State Machines

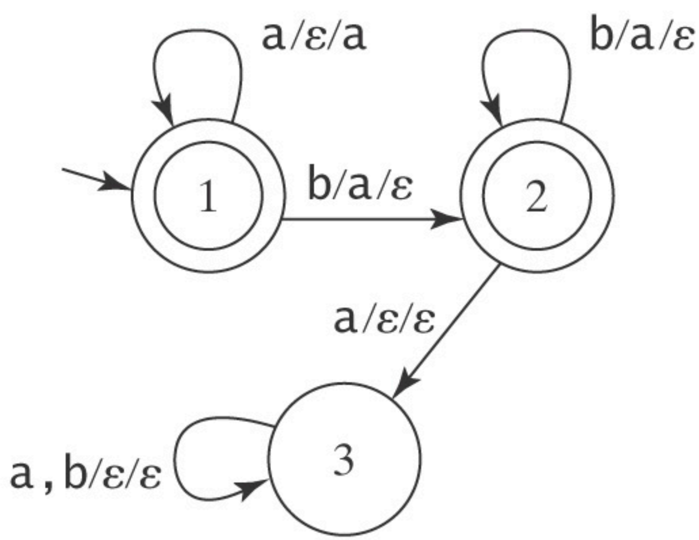
An FSM to accept a*b*:



- We call the class of languages acceptable by some FSM ***regular***
- There are simple useful languages that are not regular:
  - An FSM to accept $A^nB^n = \{a^nb^n : n \geq 0\}$
  - How can we compare numbers of a's and b's?
  - The only memory in an FSM is in the states and we must choose a fixed number of states in building it. But no bound on number of a's

Pushdown Automata

Build a PDA (roughly, FSM + a single stack)
to accept $A^nB^n = \{a^n b^n : n \geq 0\}$



Example: aaabb

Stack:

Another Example

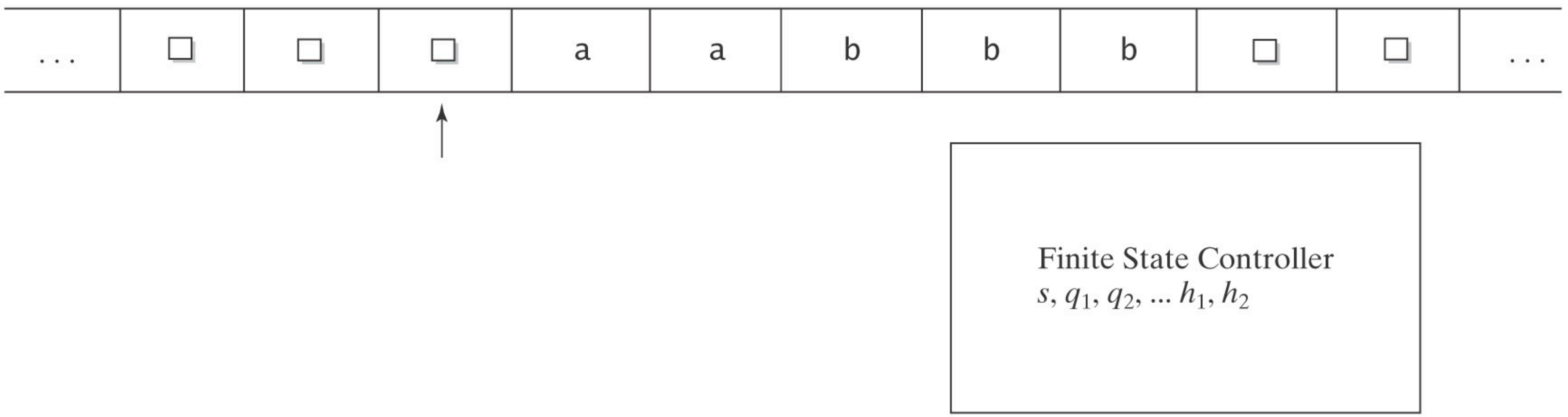
- Bal, the language of balanced parentheses
  - contains strings like (()) or ()(), but not ()))(
  - important, almost all programming languages allow parentheses, need checking
  - PDA can do the trick, not FSM

- We call the class of languages acceptable by some PDA ***context-free***.

- There are useful languages not context free.
  - $A^nB^nC^n = \{a^n b^n c^n : n \geq 0\}$
  - a stack wouldn't work. All popped out and get empty after counting b

Turing Machines

A Turing Machine to accept $A^nB^nC^n$:

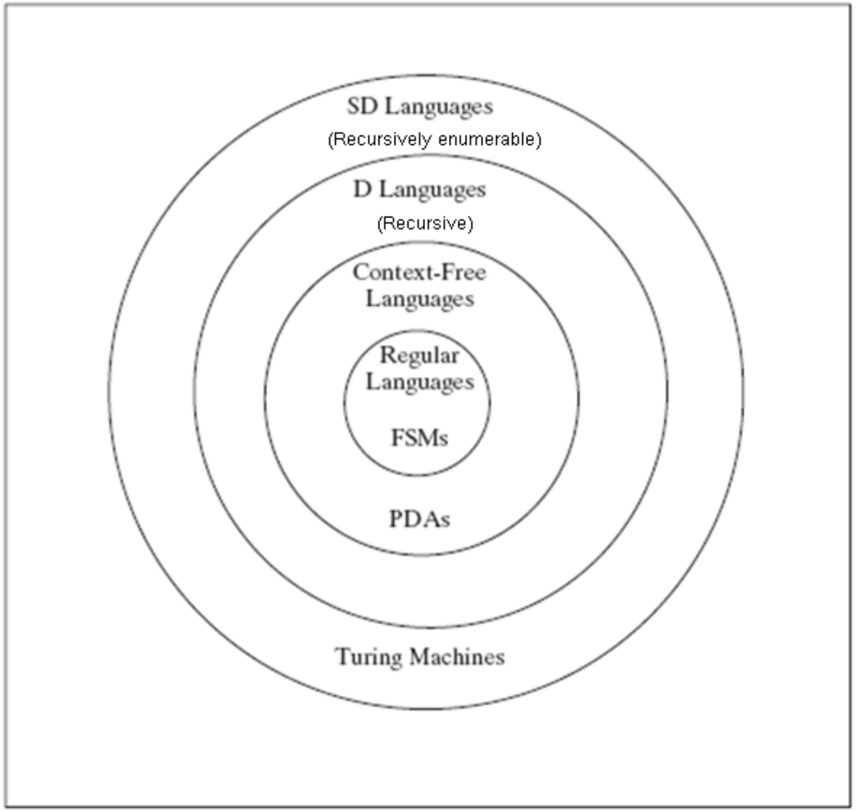


Turing Machines

- FSM and PDA (exists some equivalent PDA) are guaranteed to halt.
- But not TM. Now use TM to define new classes of languages, *D* and *SD*
- A language *L* is in *D* iff there exists a TM *M* that halts on all inputs, accepts all strings in *L*, and rejects all strings not in *L*.
    - in other words, *M* can always say yes or no properly
- A language *L* is in *SD* iff there exists a TM *M* that accepts all strings in *L* and fails to accept every string not in *L*. Given a string not in *L*, *M* may reject or it may loop forever (no answer).
    - in other words, *M* can always say yes properly, but not no.
        - give up looking? say no?
    - $D \subset SD$

- Bal, A^nB^n, $A^nB^nC^n$ … are all in *D*
    - how about regular and context-free languages?
- In *SD* but *D*: H = {<*M, w*> : TM *M* halts on input string *w*}
- Not even in *SD*: H_{all} = {<*M*> : TM *M* halts on all inputs}

Languages and Machines

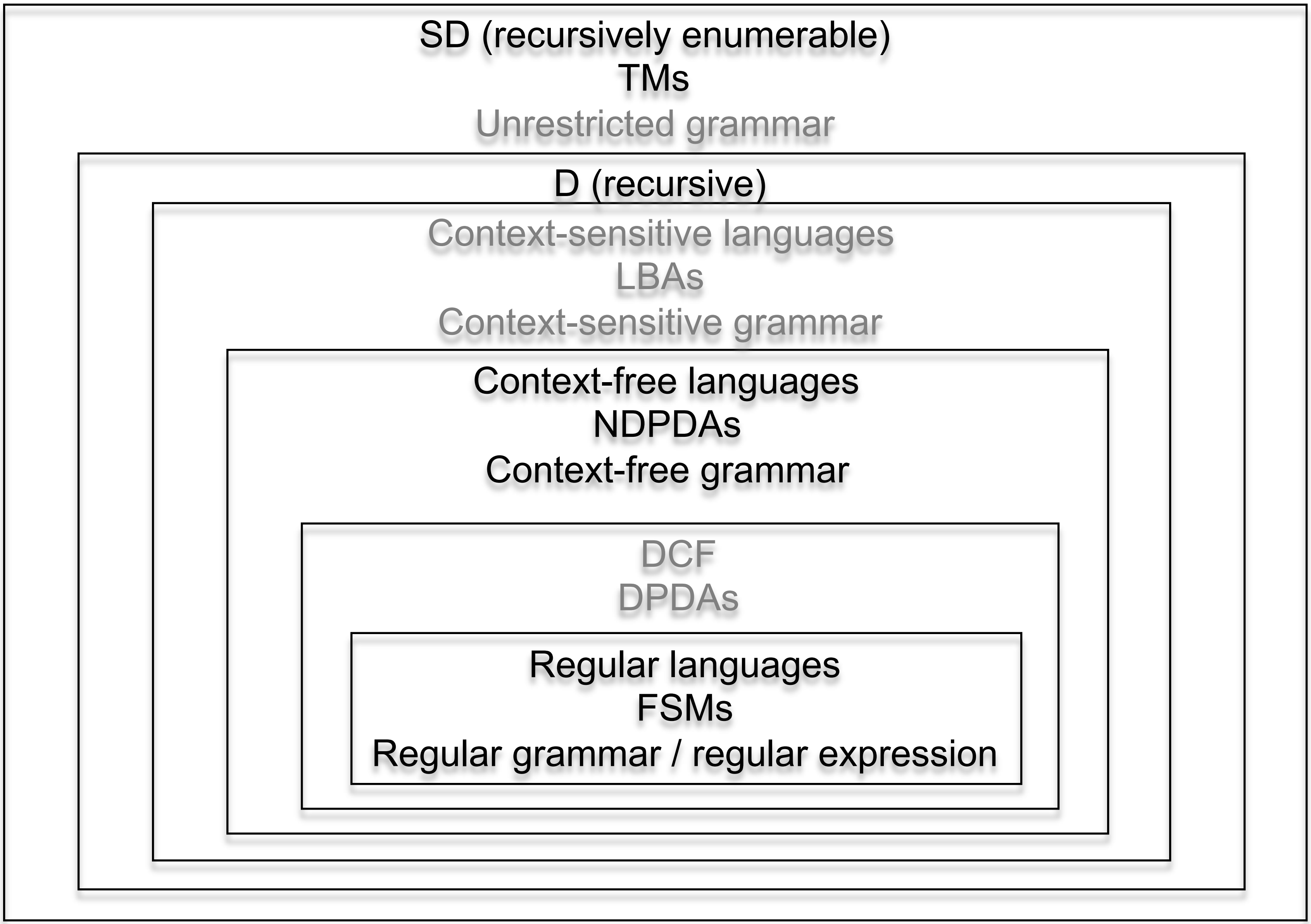


Hierarchy of language classes

Rule of Least Power:
"Use the least powerful language suitable for expressing information, constraints or programs on the World Wide Web."

- Applies far more broadly.
- Expressiveness generally comes at a price
  - computational efficiency, decidability, clarity

Languages, Machines, and Grammars



A Tractability Hierarchy

- **P** : contains languages that can be decided by a TM in polynomial time
- **NP** : contains languages that can be decided by a nondeterministic TM (one can conduct a search by guessing which move to make) in polynomial time
- **PSPACE**: contains languages that can be decided by a machine with polynomial space

$P \subseteq NP \subseteq PSPACE$

- P = NP ? Biggest open question for theorists

Decision Procedures

Chapter 4

Decidability Issues

Goal of the book: be able to make useful claims about problems and the programs that solve them.

- cast problems as language recognition tasks
- define programs as state machines whose input is a string and output is *Accept* or *Reject*

Decision Procedures

An ***algorithm*** is a detailed procedure that accomplishes some clearly specified task.

A ***decision procedure*** is an algorithm to solve a decision problem.

Decision procedures are programs and must possess two correctness properties:

- must halt on all inputs
- when it halts and returns an answer, it must be the correct answer for the given input

Decidability

- A decision problem is ***decidable*** iff there exists a decision procedure for it.
- A decision problem is ***undecidable*** iff there exists no a decision procedure for it.
- A decision problem is ***semiecidable*** iff there exists a semidecision procedure for it.
  - a semidecision procedure is one that halts and returns *True* whenever *True* is the correct answer. When *False* is the answer, it may either halt and return *False* or it may loop (no answer).
- Three kinds of problems:
  - decidable (recursive)
  - not decidable but semidecidable (recursively enumerable)
  - not decidable and not even semidecidable

Decidable

Checking for even numbers: Is the integer *x* even?

Let / perform truncating integer division, then consider the following program:

even(*x*:integer)=
    If(*x*/2)*2 = *x* then return *True* else return *False*

Is the program a decision procedure?

Undecidable but Semidecidable

Halting Problem: For any Turing machine *M* and input *w*, decide whether *M* halts on *w*.

- *w* is finite
- H = {<*M, w*> : TM *M* halts on input string *w*}
- asks whether *M* enters an infinite loop for a particular input *w*

Java version: Given an arbitrary Java program *p* that takes a string *w* as an input parameter. Does *p* halt on some particular value of *w*?

haltsOnw(*p*:program, *w*:string) =

1. simulate the execution of *p* on *w*.
2. if the simulation halts return *True* else return *False*.

Is the program a decision procedure?

Not even Semidecidable

Halting-on-all (totality) Problem: For any Turing machine *M,* decide whether *M* halts on all inputs.

- H_{ALL} = {<*M*> : TM *M* halts on all inputs}
- If it does, it computes a total function
- equivalent to the problem of whether a program can ever enter an infinite loop, for any input
- differs from the halting problem, which asks whether *M* enters an infinite loop for a particular input

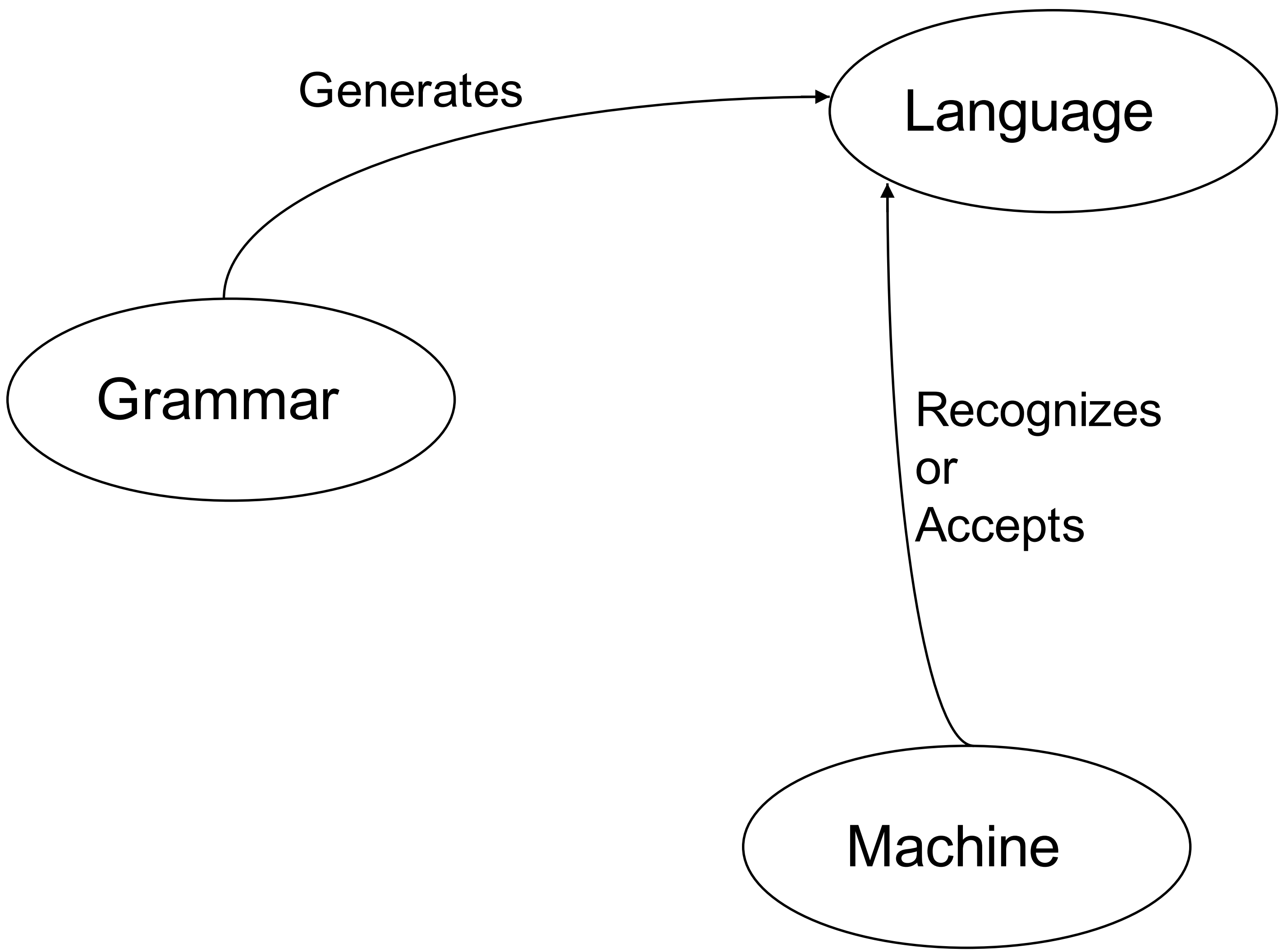
Java version: Given an arbitrary Java program *p* that takes a single string as input parameter. Does *p* halt on all possible input values?

haltsOnAll(*p*:program) =

1. for *i* = 1 to infinity do:
   simulate the execution of *p* on all possible input strings of length *i.*
2. if all the simulations halt return *True* else return *False*.

Is the program a decision procedure? A semidecision procedure?

Grammars, Languages, and Machines



Clarification

A machine M *recognizes* a language L iff *M* accepts all and only those strings in L.

A machine *M decides* a language L iff *M* accepts all strings in L and rejects all strings not in L.

recognize = accept = semi-decide ≠ decide

When a machine halts, it must either accepts or rejects. So for machines that always halt, accept implies decide.

A language L is called semi-decidable iff some TM accepts L.
A language L is called decidable iff some TM decides L.

SD: set of semi-decidable languages
D: set of decidable languages (a subset of SD by definition. actually a proper subset of SD by proof)