Examining Computational Problems

• We want to examine a given computational problem and see how difficult it is.
• Then we need to compare problems
• Problems appear different

• We want to cast them into the same kind of problem
  • decision problems
  • in particular, language recognition problem
Decision Problems

A decision problem is simply a problem for which the answer is yes or no (True or False). A decision procedure answers a decision problem.

Examples:

• Given an integer \( n \), does \( n \) have a pair of consecutive integers as factors?

• The language recognition problem: Given a language \( L \) and a string \( w \), is \( w \) in \( L \)?

Our focus
The Power of Encoding

• For problem already stated as decision problems.
  • encode the inputs as strings and then define a language that contains exactly the set of inputs for which the desired answer is yes.

• For other problems, must first reformulate the problem as a decision problem, then encode it as a language recognition task.
Everything is a String

Pattern matching on the web:

• Problem: Given a search string $w$ and a web document $d$, do they match? In other words, should a search engine, on input $w$, consider returning $d$?

• The language to be decided: $\{<w, d> : d \text{ is a candidate match for the query } w\}$
Everything is a String

Does a program always halt?

• Problem: Given a program $p$, written in some standard programming language, is $p$ guaranteed to halt on all inputs?

• The language to be decided:

$$\text{HP}_{\text{ALL}} = \{p : p \text{ halts on all inputs}\}$$
Everything is a String

What If we’re not working with strings?

Anything can be encoded as a string.

\(<X>\) is the string encoding of \(X\).
\(<X, Y>\) is the string encoding of the pair \(X, Y\).
Everything is a String

Primality Testing

• Problem: Given a nonnegative integer $n$, is it prime?
• An instance of the problem: Is 9 prime?
• To encode the problem we need a way to encode each instance: We encode each nonnegative integer as a binary string.
• The language to be decided:

$$\text{PRIMES} = \{w : w \text{ is the binary encoding of a prime number}\}.$$
Everything is a String

• Problem: Given an undirected graph $G$, is it connected?

• Instance of the problem:

```
1——— 2——— 3
     \  / \\
      4 5
```

• Encoding of the problem: Let $V$ be a set of binary numbers, one for each vertex in $G$. Then we construct $\langle G \rangle$ as follows:
  • Write $|V|$ as a binary number,
  • Write a list of edges,
  • Separate all such binary numbers by “/”.

```
101/1/10/10/11/1/100/10/101
```

• The language to be decided: $\text{CONNECTED} = \{ w \in \{0, 1, /\}^* : w = n_1/n_2/\ldots/n_i, \text{where each } n_i \text{ is a binary string and } w \text{ encodes a connected graph, as described above} \}$. 

Casting multiplication as decision:

• Problem: Given two nonnegative integers, compute their product.

• Reformulation: Transform computing into verification.

• The language to be decided:

\[ L = \{ w \text{ of the form: }<\text{integer}_1> \times <\text{integer}_2> = <\text{integer}_3>, \text{ where: }\]
\[ \text{integer}_n \text{ is any well formed integer, and } \]
\[ \text{integer}_3 = \text{integer}_1 \times \text{integer}_2 \} \]

\[ 12 \times 9 = 108 \]
\[ 12 = 12 \]
\[ 12 \times 8 = 108 \]
Turning Problems Into Decision Problems

Casting sorting as decision:

• Problem: Given a list of integers, sort it.

• Reformulation: Transform the sorting problem into one of examining a pair of lists.

• The language to be decided:

$L = \{ w_1 \neq w_2 : \exists n \geq 1 \}
(w_1 \text{ is of the form } <\text{int}_1, \text{int}_2, \ldots \text{int}_n>, \text{ \ w_2 is of the form } <\text{int}_1, \text{int}_2, \ldots \text{int}_n>, \text{ and } w_2 \text{ contains the same objects as } w_1 \text{ and } w_2 \text{ is sorted})$

Examples:

1, 5, 3, 9, 6#1, 3, 5, 6, 9
1, 5, 3, 9, 6#1, 2, 3, 4, 5, 6, 7
Turning Problems Into Decision Problems

Casting database querying as decision:

• Problem: Given a database and a query, execute the query.

• Reformulation: Transform the query execution problem into evaluating a reply for correctness.

• The language to be decided:

\[ L = \{ d \# q \# a : \]
\[ d \text{ is an encoding of a database,} \]
\[ q \text{ is a string representing a query, and} \]
\[ a \text{ is the correct result of applying } q \text{ to } d \} \]

Example:

(name, age, phone), (John, 23, 567-1234)
(Mary, 24, 234-9876)#(select name age=23)#(John)
The Traditional Problems and their Language Formulations are Equivalent

By equivalent we mean that either problem can be reduced to the other.

If we have a machine to solve one, we can use it to build a machine to do the other using just the starting machine and other functions that can be built using a machine of equal or lesser power.
An Example

Consider the multiplication example:

\[ L = \{w \text{ of the form:} \]
\[<\text{integer}_1> \times <\text{integer}_2> = <\text{integer}_3>, \text{ where:} \]
\[\text{integer}_n \text{ is any well formed integer, and} \]
\[\text{integer}_3 = \text{integer}_1 \times \text{integer}_2}\]

Given a multiplication machine, we can build the language recognition machine:

Given the language recognition machine, we can build a multiplication machine:
Languages and Machines

- SD Languages (Recursively enumerable)
- D Languages (Recursive)
- Context-Free Languages
- Regular Languages
- FSMs
- PDAs
- Turing Machines
Finite State Machines

An FSM to accept $a^*b^*$:

- We call the class of languages acceptable by some FSM regular.
- There are simple useful languages that are not regular:
  - An FSM to accept $A^nB^n = \{a^n b^n : n \geq 0\}$
  - How can we compare numbers of a’s and b’s?
  - The only memory in an FSM is in the states and we must choose a fixed number of states in building it. But no bound on number of a’s
Pushdown Automata

Build a PDA (roughly, FSM + a single stack) to accept $A^n B^n = \{a^n b^n : n \geq 0\}$

Example: $aaabbb$

Stack:
Another Example

• Bal, the language of balanced parentheses
  • contains strings like (()) or ()(), but not ()())
  • important, almost all programming languages allow parentheses, need checking
  • PDA can do the trick, not FSM

• We call the class of languages acceptable by some PDA context-free.

• There are useful languages not context free.
  • $A^nB^nC^n = \{a^nb^nC^n : n \geq 0\}$
  • a stack wouldn’t work. All popped out and get empty after counting b
Turing Machines

A Turing Machine to accept $A^nB^nC^n$:

| ... | □ | □ | □ | a | a | b | b | b | □ | □ | ... |

Finite State Controller
$s, q_1, q_2, ... h_1, h_2$
Turing Machines

- FSM and PDA (exists some equivalent PDA) are guaranteed to halt.
- But not TM. Now use TM to define new classes of languages, $D$ and $SD$
- A language $L$ is in $D$ iff there exists a TM $M$ that halts on all inputs, accepts all strings in $L$, and rejects all strings not in $L$.
  - in other words, $M$ can always say yes or no properly
- A language $L$ is in $SD$ iff there exists a TM $M$ that accepts all strings in $L$ and fails to accept every string not in $L$. Given a string not in $L$, $M$ may reject or it may loop forever (no answer).
  - in other words, $M$ can always say yes properly, but not no.
  - give up looking? say no?
- $D \subset SD$
- Bal, $A^nB^n$, $A^nB^nC^n$ … are all in $D$
  - how about regular and context-free languages?
- In $SD$ but $D$: $H = \{<M, w> : TM \text{ } M \text{ halts on input string } w\}$
- Not even in $SD$: $H_{all} = \{<M> : TM \text{ } M \text{ halts on all inputs}\}$
Rule of Least Power:
“Use the least powerful language suitable for expressing information, constraints or programs on the World Wide Web.”

- Applies far more broadly.
- Expressiveness generally comes at a price
  - computational efficiency, decidability, clarity
Languages, Machines, and Grammars

- SD (recursively enumerable)
- TMs
- Unrestricted grammar

- D (recursive)
- Context-sensitive languages
- LBAs
- Context-sensitive grammar

- Context-free languages
- NDPDAs
- Context-free grammar

- DCF
- DPDAs

- Regular languages
- FSMs
- Regular grammar / regular expression
Grammars, Languages, and Machines

Grammar \( \rightarrow \) Language

Generates

Language \( \rightarrow \) Machine

Recognizes or Accepts
A Tractability Hierarchy

- **P**: contains languages that can be decided by a TM in polynomial time

- **NP**: contains languages that can be decided by a nondeterministic TM (one can conduct a search by guessing which move to make) in polynomial time

- **PSPACE**: contains languages that can be decided by a machine with polynomial space

\[ P \subseteq NP \subseteq PSPACE \]

- **P = NP?** Biggest open question for theorists
Decision Procedures

Chapter 4
Decidability Issues

Goal of the book: be able to make useful claims about problems and the programs that solve them.
• cast problems as language recognition tasks
• define programs as state machines whose input is a string and output is *Accept or Reject*
An **algorithm** is a detailed procedure that accomplishes some clearly specified task.

A **decision procedure** is an algorithm to solve a decision problem.

Decision procedures are programs and must possess two correctness properties:

- must halt on all inputs
- when it halts and returns an answer, it must be the correct answer for the given input
Decidability

• A decision problem is **decidable** iff there exists a decision procedure for it.

• A decision problem is **undecidable** iff there exists no a decision procedure for it.

• A decision problem is **semidecidable** iff there exists a semidecision procedure for it.
  • a semidecision procedure is one that halts and returns *True* whenever *True* is the correct answer. When *False* is the answer, it may either halt and return *False* or it may loop (no answer).

• Three kinds of problems:
  • decidable (recursive)
  • not decidable but semidecidable (recursively enumerable)
  • not decidable and not even semidecidable

• **Note:** Usually defined w.r.t. Turing machines
  • most powerful formalism for algorithms
  • decidable = Turing-decidable
Checking for even numbers: Is the integer $x$ even?

Let / perform truncating integer division, then consider the following program:

$$\text{even}(x:\text{integer})=\begin{cases} \text{True} & \text{if } (x/2)\times2 = x \\ \text{False} & \text{otherwise} \end{cases}$$

Is the program a decision procedure?
Undecidable but Semidecidable

Halting Problem: For any Turing machine $M$ and input $w$, decide whether $M$ halts on $w$.
- $w$ is finite
- $H = \{<M, w> : \text{TM } M \text{ halts on input string } w\}$
- asks whether $M$ enters an infinite loop for a particular input $w$

Java version: Given an arbitrary Java program $p$ that takes a string $w$ as an input parameter. Does $p$ halt on some particular value of $w$?

$\text{haltsOnw}(p:\text{program}, w:\text{string}) =$
1. simulate the execution of $p$ on $w$.
   2. if the simulation halts return $True$ else return $False$.

Is the program a decision procedure?
Halting-on-all (totality) Problem: For any Turing machine $M$, decide whether $M$ halts on all inputs.

- $H_{\text{ALL}} = \{<M>: \text{TM } M \text{ halts on all inputs}\}$
- If it does, it computes a total function
- equivalent to the problem of whether a program can ever enter an infinite loop, for any input
- differs from the halting problem, which asks whether $M$ enters an infinite loop for a particular input

Java version: Given an arbitrary Java program $p$ that takes a single string as input parameter. Does $p$ halt on all possible input values?

```java
haltsOnAll(p:program) =
  1. for $i = 1$ to infinity do:
     simulate the execution of $p$ on all possible input strings of length $i$.
  2. if all the simulations halt return True else return False.
```

Is the program a decision procedure? A semidecision procedure?