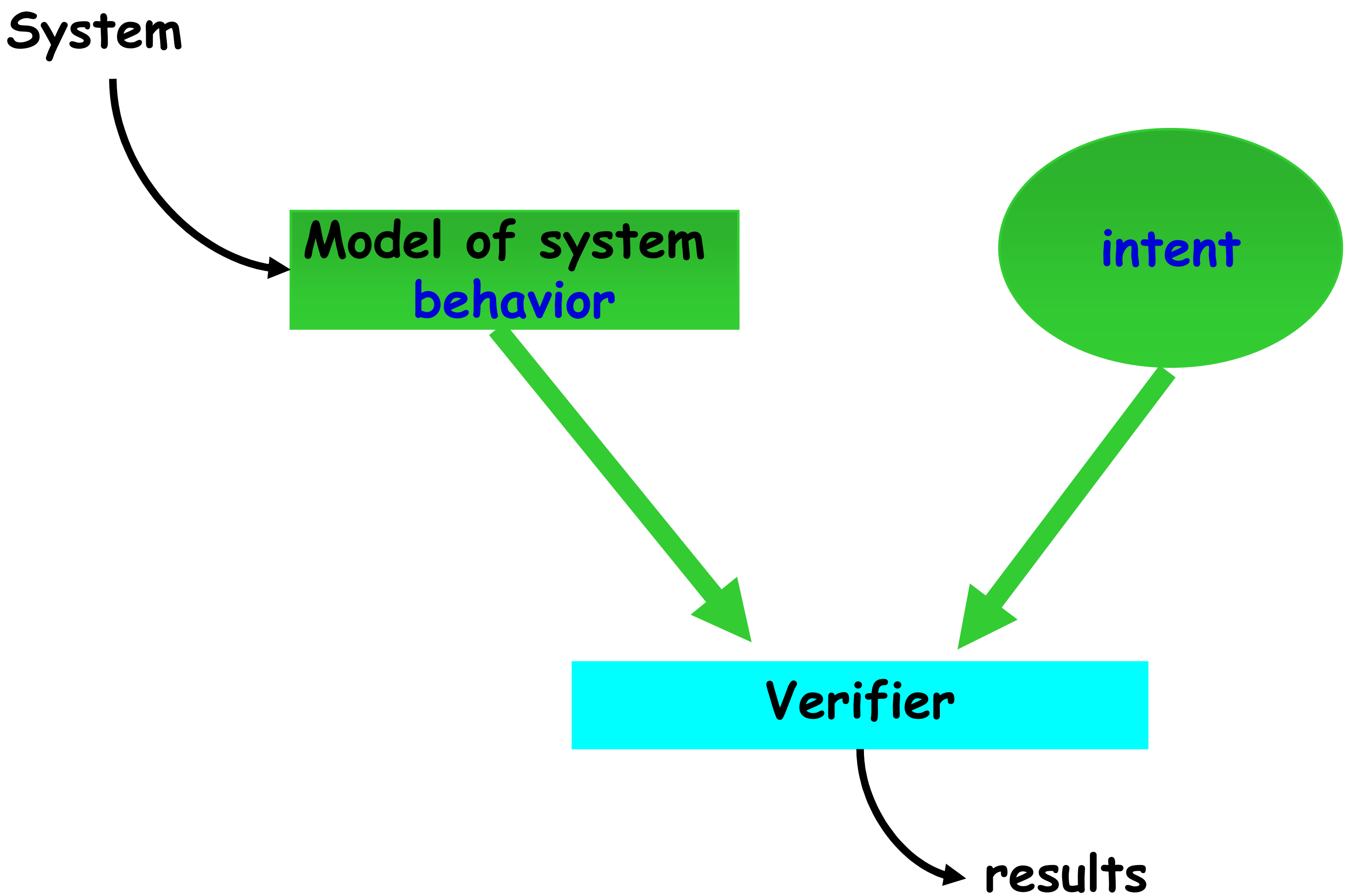
Formal Verification

Basic Verification Strategy

compare behavior to intent



Intent

- Usually, originates with requirements, refined through design and implementation
- formalized by specifications
  - Often expressed as formulas in mathematical logic
- different types of intent
  - E.g.,performance, functional behavior
  - each captured with different types of formalisms
  - specification of behavior/functionality
    - what functions does the software compute?
    - Often expressed using predicate logic

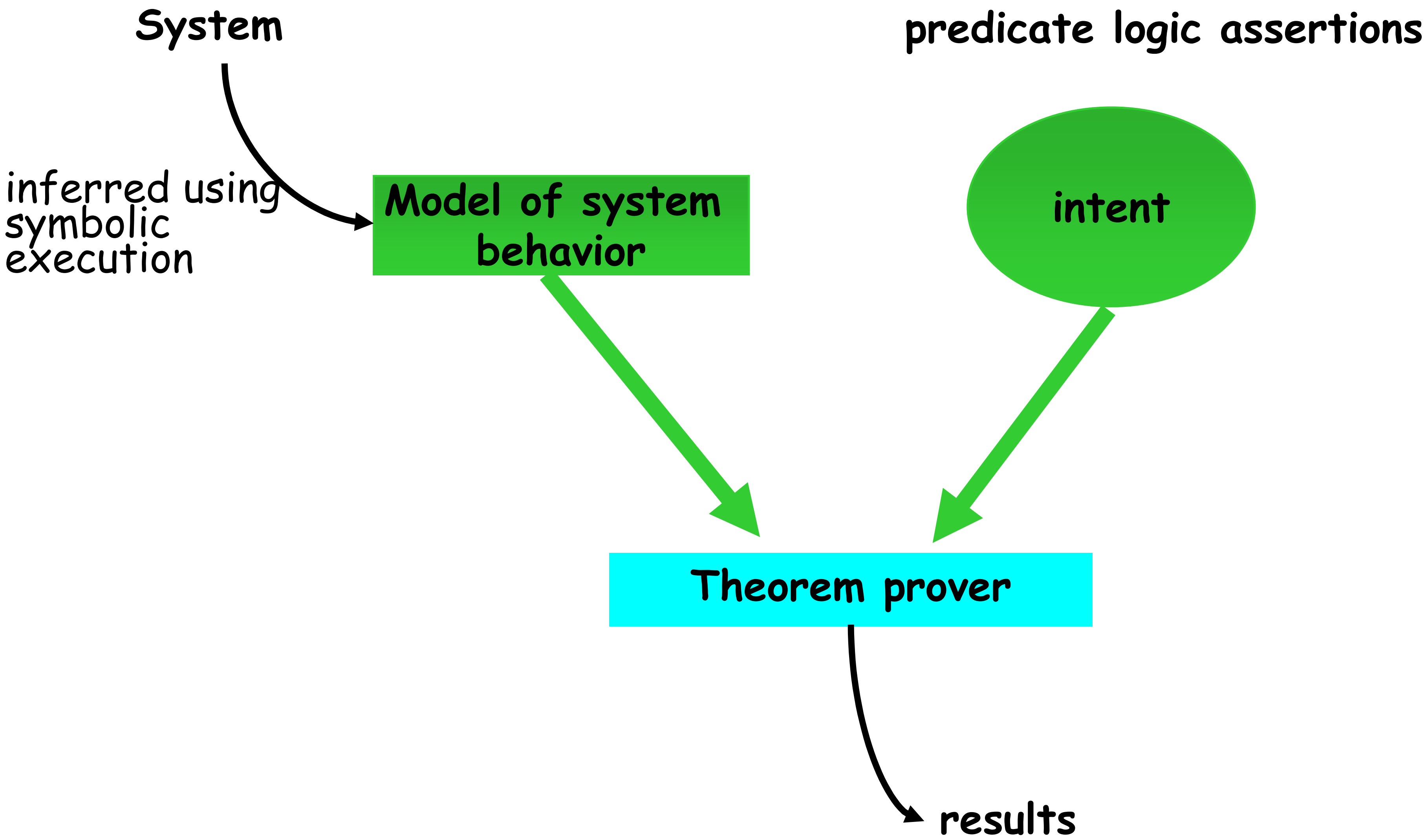
Compare behavior to intent

- can be done informally- by human eye
  - Cleanroom
  - Inspections
- can be done selectively
  - Checking assertions during execution
- can be done formally
  - With theorem proving
    - Usually with automated support
    - Called Proof of Correctness or Formal Verification
      - Proof of "correctness" is dangerously misleading
  - With static analysis for restricted classes of properties

Theorem Proving based Verification

- **Behavior inferred from semantically rich program model**
  - **generally requires most of the semantics of the programming language**
  - **employs symbolic execution**
- **Intent captured by predicate calculus specifications (or another mathematically formal notation)**

Theorem-Proving based Verification Strategy



Floyd Method of Inductive Assertions

- Show that given input assertions, after executing the program, program satisfies output assertions
  - show that each program fragment behaves as intended
  - use induction to prove that all fragments, including loops, behave as intended
- show that the program must terminate

Mathematical Induction

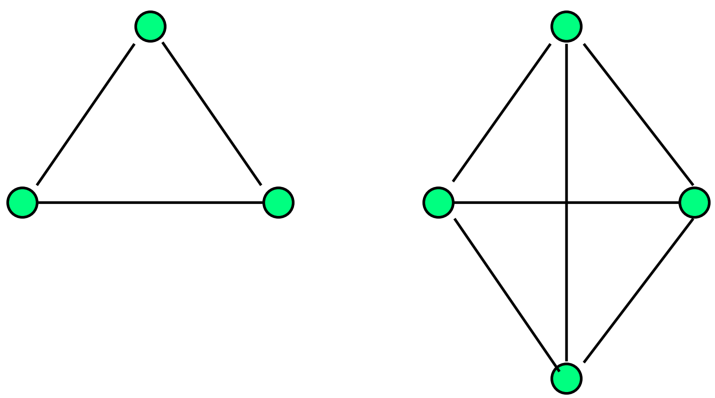
- goal: prove that a given property holds for all elements of a set
- approach:
  - show property holds for "first" element
  - show that if property holds for element i, then it must also hold for element i + 1
- often used when direct analytic techniques are too hard or complex

Example: How many edges in C_n

Theorem:

let $C_n = (V_n, E_n)$ be a complete, unordered graph on n nodes,

then $|E_n| = n * (n-1)/2$



Example: How many edges in C_n

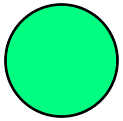
- to show that this property holds for the entire set of complete graphs, $\{C_i\}$, by induction:

    1. show the property is true for C_1
    2. show if the property is true for C_n ,then the property is true for C_{n+1}

Example: How many edges in C_n

show the property is true for C1:

graph has one node, 0 edges



$$|E_1| = n(n-1)/2 = 1(0)/2 = 0$$

Example: How many edges in C_n

assume true for C_n: $|E_n| = n(n-1)/2$

graph C_{n+1} has one more node, but n more edges (one from the new node to each of the n old nodes)

Thus, want to show $|E_{n+1}| = |E_n|+n =(n+1)(n)/2$

Proof: $|E_{n+1}| = |E_n|+n = n(n-1)/2 + n$ by substitution

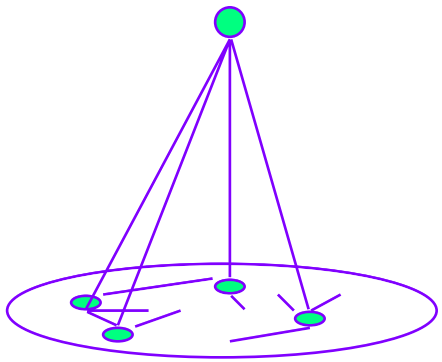
$= n(n-1)/2 +2n/2$ by rewriting

$= (n(n-1)+2n)/2$ by simplification

$= (n(n-1+2))/2$ by simplification

$= n(n+1)/2$ by simplification

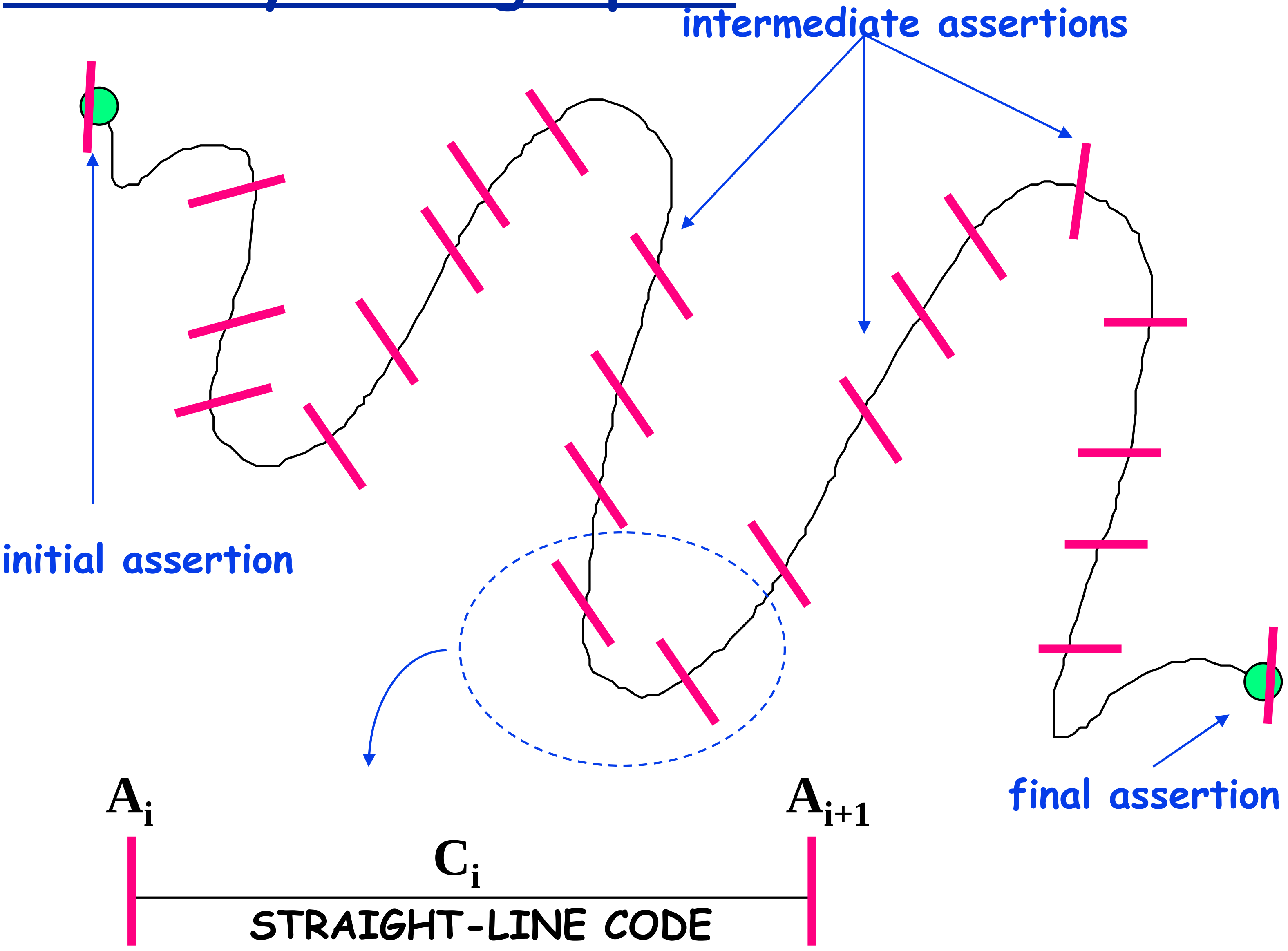
$= (n+1)(n)/2$ by rewriting



Floyd's Method of inductive verification (informal description)

- Place assertions at the start, final, and intermediate points in the code.
- Any path is composed of sequences of program fragments that start with an assertion, are followed by some assertion free code, and end with an assertion
  - $A_s, C_1, A_2, C_2, A_3, \ldots A_{n-1}, C_{n-1}, A_f$
- Show that for every executable path, if A_s is assumed true and the code is executed, then A_f is true

Pictorially: A single path

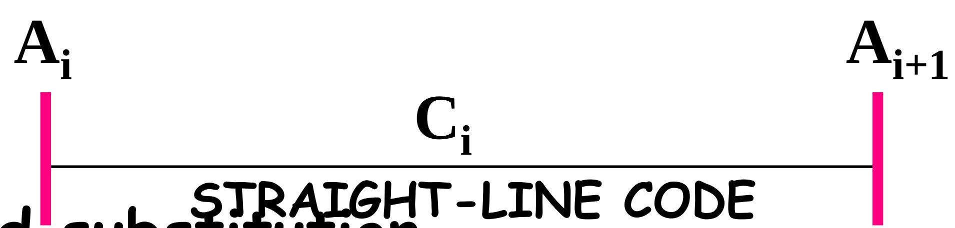


Must be sure:

assuming A_i,

then executing Code C_i,

necessarily $\Rightarrow$ $A_i + 1$



by forward substitution

$\Rightarrow$ symbolic execution

Why does this work?

suppose P is an arbitrary path through the program

can denote it by

$$P = A_0\ C_1\ A_1\ C_2\ A_2 \ldots C_n\ A_n$$

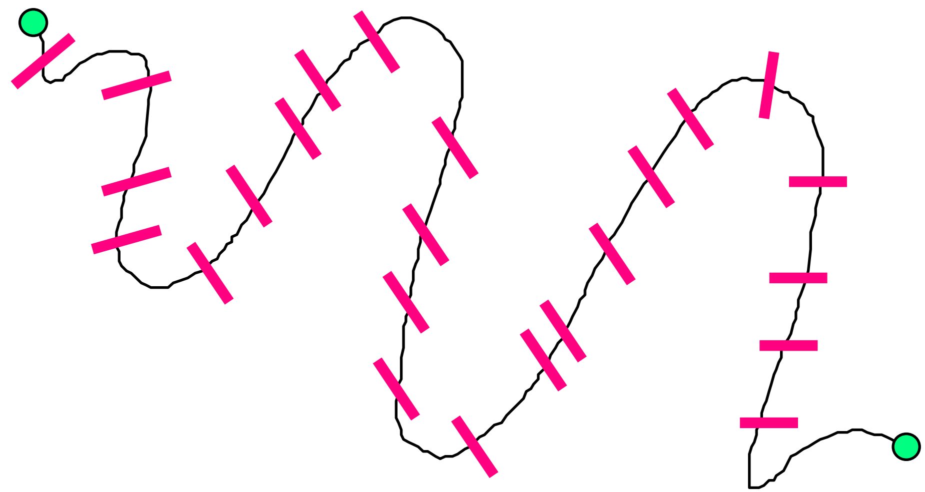
where

A_0 - Initial assertion

A_n - Final assertion

A_i - Intermediate assertions

C_i - Loop free, uninterrupted, straight-line code



If it has been shown that

$\forall$ i, 1 $\leq$ i < n: $A_i C_i \Rightarrow A_{i+1}$

Then, by transitivity

$A_0 \ldots\ldots \Rightarrow A_n$

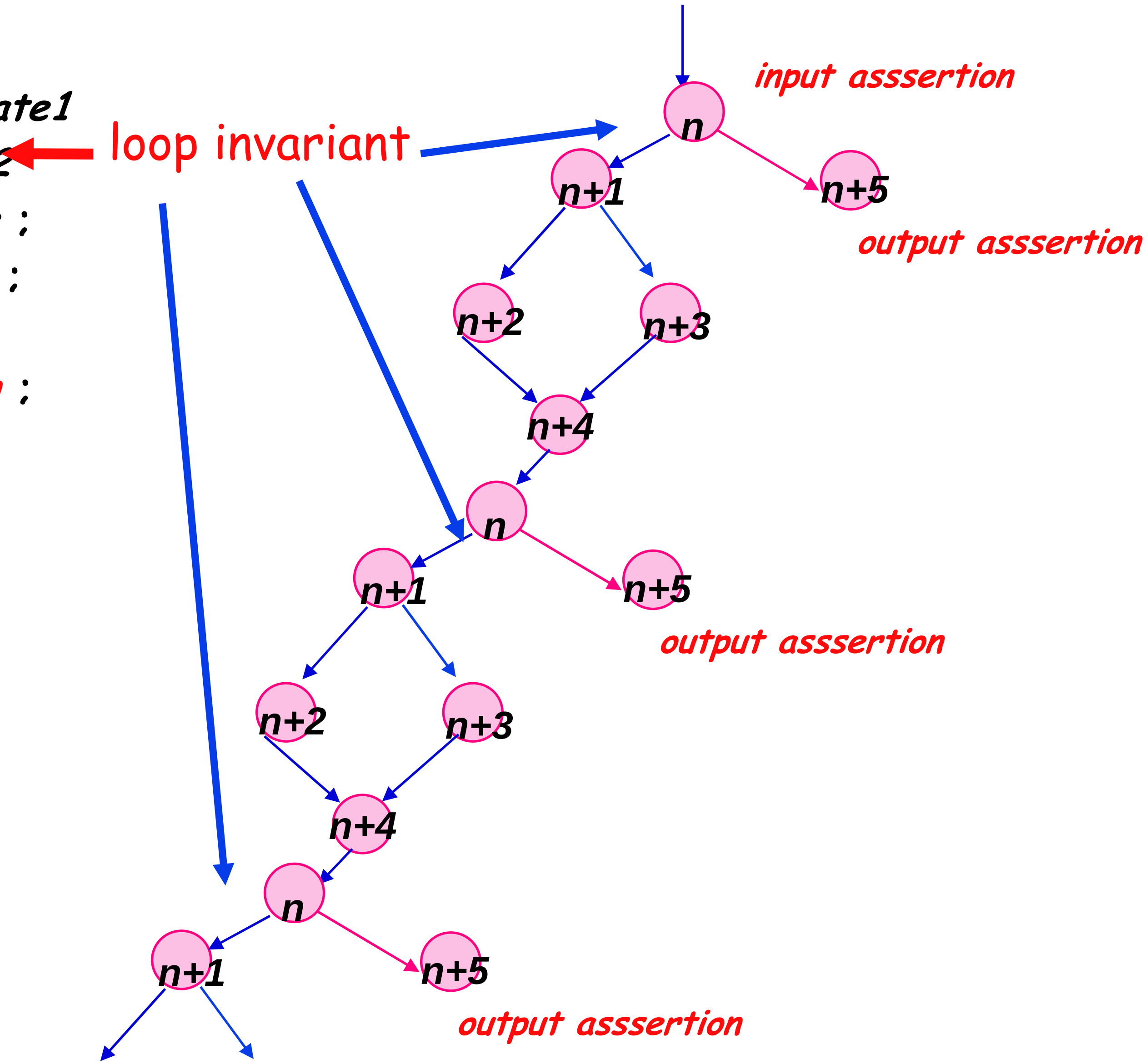
Obvious problems

- How do we do this for a path?
- How do we do this for all paths?
  - Infinite number of paths
    - Must find a way to deal with loops

How to handle loops -- unroll them

input asssertion

n	do_while *predicate1*
n+1	if *predicate2*
n+2	then *code* ;
n+3	else *code* ;
n+4	end;
n+5	output *assertion* ;



Better -- find loop invariant (A_I)

subpaths to consider:

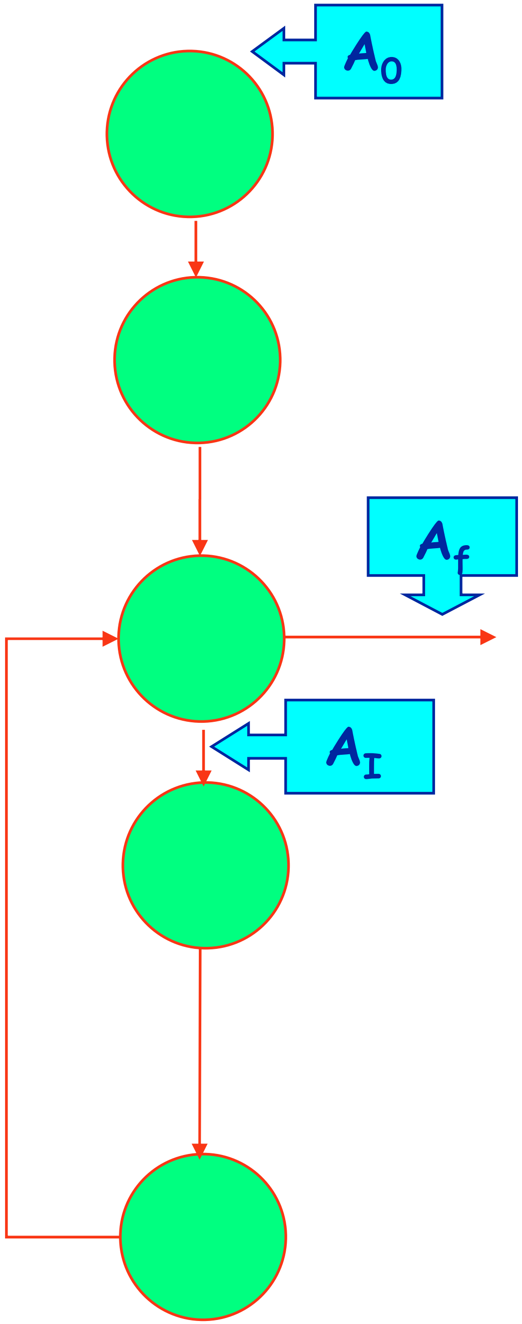
$C_{1:}$ **Initial assertion A_0 to final assertion A_f**

$C_{2:}$ **Initial assertion A_0 to A_I**

$C_{3:}$ **A_I to A_I**

$C_{4:}$ **A_I to final assertion A_f**

Basically an inductive proof



Consider all paths through a loop

subpaths to consider:

$C_{1:}$ A_0 to A_f

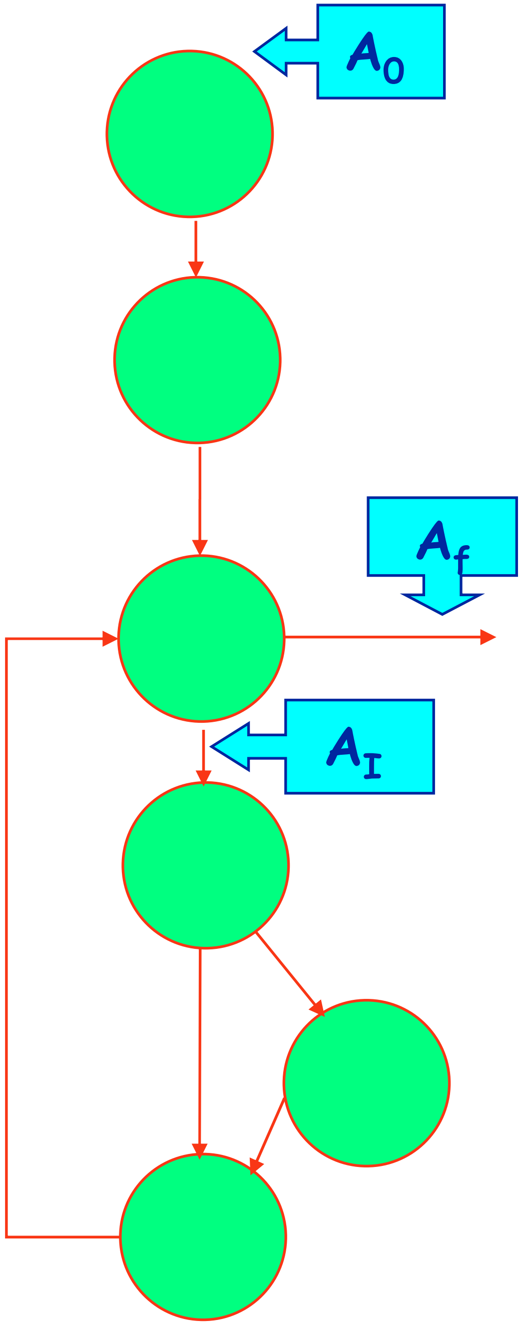
$C_{2:}$ A_0 to A_I

$C_{3:}$ A_I, false branch, A_I

$C_{4:}$ A_I, true branch, A_I

$C_{5:}$ A_I, false branch, A_f

$C_{6:}$ A_I, true branch, A_f



Assertions

- specification that is intended to be true at a given site in the program
- Use three types of assertions:
  - initial : sited before the initial statement
  - final : sited after the final statement
  - intermediate: sited at various internal program locations subject to the rule:
    - every loop iteration shall pass through the site of at least one intermediate assertion
    - a "loop invariant" is true on every iteration thru the loop

Floyd's Inductive Verification Method (more carefully stated)

- specify initial and final assertions to capture intent
- place intermediate assertions so as to "cut" every program loop
- For each pair of assertions where there is at least one executable (assertion-free) path from the first to the second,
  - assume that the first assertion is true
  - show that for all (assertion-free, executable) paths from the first assertion to the second, that the second assertion is true
  - This establishes "partial correctness"
- Show that the program terminates
  - This establishes "total correctness"

Floyd-Hoare axiomatic proof method

assertions are preconditions and postconditions on some statement or sequence of statements

$$P\{S\}Q$$

if P is true before S is executed and S is executed then Q is true

P is the precondition; Q is the postcondition

Also written {P} S {Q}

Floyd-Hoare axiomatic proof method

- as in Floyd's inductive assertion method, we construct a sequence of assertions, each of which can be inferred from previously proved assertions and the rules and axioms about the statements and operations of the program
- to prove P{S}Q, we need some axioms and rules about the programming language

Hoare axioms and proof rules

take a simple programming language that deals only with integers and has the following types of constructs:

- assignment statement

  x:= f
- composition of a sequence of statements

  S1, S2
- conditional (alternative statements)

  if B then S1 else S2
- iteration

  while B do S

Axioms and proof rules

- **axiom of assignment**

  P {x:=f} Q,

  where Q is obtained from P by substituting f for all occurences of x in P (symbolic execution)

- **rule of composition**

  P {S1, S2 } Q => ∃ P1 , P{S1}P1 Λ P1{S2}Q

  Using Hoare's notation, this is written as

$$\frac{P\{S1\}P1,\ P1\{S2\}Q}{P\ \{S1, S2\}\ Q}$$

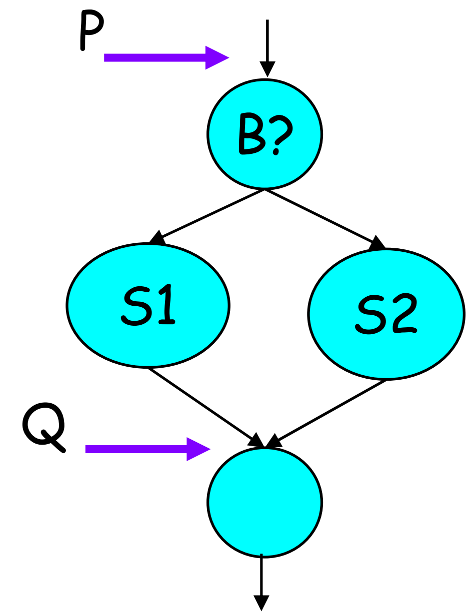
Proof Rules (continued)

- rule for the alternative statement

  $P\{\text{if } B \text{ then } S1 \text{ else } S2\}Q \Rightarrow$

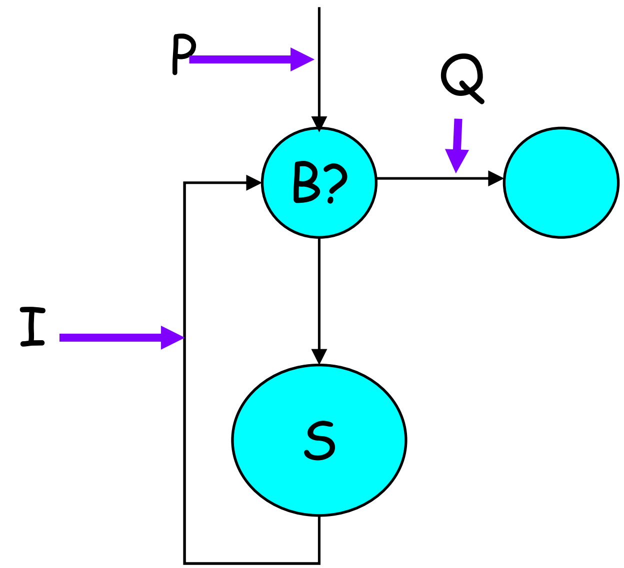
  $P\{B \wedge S1\}Q \wedge P\{\sim B \wedge S2\}Q$

- Hoare's notation



$$\frac{P\{B \wedge S1\}Q,\ P\{\sim B \wedge S2\}Q}{P\{\text{if } B \text{ then } S1 \text{ else } S2\}Q}$$

Proof Rules (continued)



rule of iteration

$$P\ \{\text{while } B \text{ do } S\}\ Q \Rightarrow$$
$$P\{\sim B\}Q \wedge \exists I \ni P\ \{B \wedge S\}\ I$$
$$\wedge\ I\{B \wedge S\}\ I \wedge I\{\sim B\}Q$$

$$\frac{P\{\sim B\}Q\ ,P\ \{B \wedge S\}\ I\ ,\ I\{B \wedge S\}\ I,\ I\{\sim B\}Q}{P\ \{\text{while } B \text{ do } S\}Q}$$

weakest precondition

- in Hoare technique P{S}Q

```
S1:
read x,y;
z:= y
while x >0 do
    z:= z+1;
    x:= x-1;
endwhile;
```

```
S2:
read x,y;
z:= x+y;
```

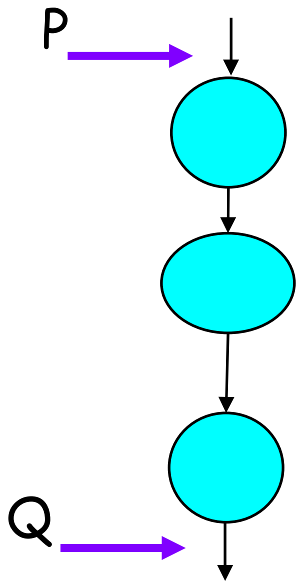
suspose $P = \{x \geq 0\}$

$Q = \{z = x+y\}$

- then we can prove P{S1}Q and P{S2}Q, but we can also prove true{S2}Q
- S2 is provable for any x, y, but S1 is provable only for $x \geq 0$

Dijkstra's Axiomatic Semantics

- In general, there are many correct pre- and post-conditions for a given program
- Seek the strongest post condition and the weakest precondition
  - $P \Rightarrow P'$; P is **stronger** than P' and P' is **weaker** than P



Rules of consequence

- If $P \Rightarrow P'$ and $Q' \Rightarrow Q$ and $P'\{S\}Q'$ then $P\{S\}Q$

$$\frac{P\{S\}Q', \; Q' \Rightarrow Q}{P\{S\}Q}$$

$$\frac{P \Rightarrow P', \; P'\{S\}Q}{P\{S\}Q}$$

$$\frac{P \Rightarrow P', \; P'\{S\}Q', \; Q' \Rightarrow Q}{P\{S\}Q}$$

Formal Verification Process

- determine input, output and loop invariant assertions
- identify all paths between two assertions (with no intervening assertions) and form the corresponding verification condition or lemma
- prove each verification condition (partial correctness)
- prove that the program terminates