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ON THE FRAME–STEWART CONJECTURE ABOUT THE TOWERS OF HANOI*

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Abstract. The multipeg Towers of Hanoi problem consists of k pegs mounted on a board together with n disks of different sizes. Initially these disks are placed on one peg in the order of their size, with the largest at the bottom. The rules of the problem allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the problem is to transfer all the disks to another peg with the minimum number of moves, denoted by $H(n, k)$. An easy recursive argument shows that $H(n, 3) = 2^n - 1$. However, the problem of computing the exact value of $H(n, k)$ for $k \geq 4$ has been open since 1939, and in particular, the special case of $H(n, 4)$ has been open since 1907.

In 1941, Frame and Stewart each gave an algorithm to solve the Towers of Hanoi problem based on an unproved assumption. The Frame–Stewart number, denoted by $FS(n, k)$, is the number of moves needed to solve the Towers of Hanoi problem using the "presumed optimal" Frame–Stewart algorithm. Since then, proving the Frame–Stewart conjecture $FS(n, k) = H(n, k)$ has become a notorious open problem.

In this paper, we prove that $FS(n, k)$ and $H(n, k)$ both have the same order of magnitude of $2^{(1\pm o(1))(n(k-2)!)^{1/(k-2)}}$. This provides the strongest evidence so far to support the Frame–Stewart conjecture.

Key words. Towers of Hanoi problem, Frame–Stewart conjecture, optimal algorithm, Frame–Stewart algorithm

AMS subject classifications. 05A10, 05A16, 68R05

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1. Introduction.

The Towers of Hanoi problem, introduced by Édouard Lucas in 1883, consists of three pegs and a set of n disks of different diameters that can be stacked on the pegs. The towers are formed initially by stacking the disks onto one peg in the order of their size, with the largest at the bottom. The rules of the problem allow disks to be moved one at a time from one peg to another as long as a disk is never placed on the top of a smaller disk. The goal of the problem is to transfer all the disks to another peg with the minimum number of moves. An easy argument using a recursive relation shows that $2^n - 1$ moves are necessary and sufficient to carry out this task.

The Towers of Hanoi problem was extended to four pegs by Dudeney [2] in 1907 and to any arbitrary $k \geq 3$ pegs by Stewart [7] in 1939. In 1941, Frame [3] and Stewart [8] independently proposed an algorithm to the Towers of Hanoi problem with $k \geq 4$ pegs:

1. Recursively transport a stack of $n - i$ smallest disks from the first peg to a temporary peg, using all k pegs;
2. Transport the remaining stack of i largest disks from the first peg to the final peg, using $k - 1$ pegs and ignoring the peg occupied by the smaller disks;

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3. Recursively transport the smallest $n-i$ disks from the temporary peg to the final peg, using all k pegs.

(Indeed, Frame's algorithm is slightly different from the above proposed by Stewart. But both algorithms are essentially equivalent [4].) The Frame–Stewart number, denoted by $FS(n,k)$, is the minimum number of moves needed to solve the Towers of Hanoi problem using the above Frame–Stewart algorithm. Thus $FS(n,k)$ has the following recursive formula:

$$FS(n,k) = \begin{cases} 2^n - 1 & \text{if } k = 3, \\ \min_{1 \le i < n}\{2FS(n-i,k) + FS(i,k-1)\} & \text{if } k \ge 4. \end{cases}$$

The Frame–Stewart number $FS(n,k)$ is called the "presumed optimal" solution since no justification has ever been made that an optimal algorithm must be of this form. Let $H(n,k)$ be the minimum number of moves needed to solve the Towers of Hanoi problem. The Frame–Stewart conjecture $FS(n,k) = H(n,k)$ is still open now. (As pointed out by Klavžar, Milutinović, and Petr [4], the claimed proof of the conjecture by Majumdar [6] is indeed incorrect.) Donald Knuth commented on the conjecture, saying "I doubt if anyone will ever resolve the conjecture; it is truly difficult" (see [5]).

In the attempt to prove the Frame–Stewart conjecture, Bode and Hinz [1] verified the conjecture for four pegs with up to 17 disks. Recently, Szegedy [9] proved that

$$FS(n,k) \ge 2^{(1\pm o(1))c_k n^{1/(k-2)}},$$

where $c_k = \frac{1}{2}\left(\frac{12}{k(k-1)}\right)^{1/(k-2)}$.

For convenience, let $\log x$ denote the logarithmic function with base 2. In this paper, we prove that for $n \ge 1$ and $k \ge 3$,

$$\log FS(n,k) = \log H(n,k) + \Theta(k + \log n) = (n(k-2)!)^{1/(k-2)} + \Theta(k + \log n).$$

In other words, for each fixed $k \ge 3$ and for $n \gg k$,

$$FS(n,k) = 2^{(1\pm o(1))(n(k-2)!)^{1/(k-2)}} = H(n,k).$$

This provides the strongest evidence so far to support the Frame–Stewart conjecture.

2. Lower bound on the optimal number of moves. In this section, we derive a lower bound on the optimal number $H(n,k)$ of moves for the Towers of Hanoi problem. We adopt the remarkable strategy introduced by Szegedy [9] who considered the following generalized problem: What is the minimum number of moves to move each disk at least once among all possible initial setups of disks? The advantage of this strategy is that one can use induction in the proofs.

An arrangement of k pegs and n disks is called a configuration if it obeys the "smaller disk on the top of larger disk" rule. For a configuration C, let $g(C)$ be the minimum number of moves required to have every disk moved at least once, where all moves are taken according to the rules of the Towers of Hanoi. Szegedy [9] defined

$$g(n,k) = \min_C g(C),$$

where C runs through all possible configurations of n disks and k pegs. The function $g(n,k)$ is well defined since $g(C)$ is finite for some configuration C [9, Remark 1]. By the definition of $g(n,k)$, we have $H(n,k) \ge g(n,k)$, and thus a lower bound on

$H(n,k)$ can be derived from Theorem 2. We begin with a study of the monotone properties of the function $g(n,k)$.

LEMMA 1. *Suppose $n \geq 1$ and $k \geq 3$. Then the function $g(n,k)$ is decreasing with respect to the variable k.*

THEOREM 1. *Suppose $n \geq 2$ and $k \geq 4$. Then there exists some m with $1 \leq m \leq n-1$ such that*

$$g(n,k) \geq \begin{cases} 2\max\{g(n-m,k), g(m,k-1)\} & \text{if } g(n-m,k) = g(m,k-1), \\ 2\max\{g(n-m,k), g(m,k-1)\} - 1 & \text{if } g(n-m,k) \neq g(m,k-1). \end{cases}$$

Proof. Let C be an extremal configuration of n disks and k pegs with $g(C) = g(n,k)$. Let $S = \big(s_1, s_2, \ldots, s_{g(C)}\big)$ be a sequence of $g(C)$ moves that move every disk of C at least once, where all moves are taken according to the rules of the Towers of Hanoi. Let $S = S_1 \cup S_2$ with $|S_1| = \lfloor g(C)/2 \rfloor$ and $|S_2| = \lceil g(C)/2 \rceil$; that is, S_1 and S_2 are the first half and the second half of the sequence of moves of S, respectively. For $i = 1, 2$, let

$$D_i = \{j : \text{disk } j \text{ is moved at least once by } S_i\}.$$

Then $|D_1 \cup D_2| = n$.

Claim. $D_1 - D_2 \neq \emptyset$ and $D_2 - D_1 \neq \emptyset$.

Proof of the claim. Let $s_1(C)$ be the configuration obtained by applying the first move s_1 of S to C. Suppose the move s_1 moves disk i. We observe that disk i cannot be moved more than once by S; otherwise, each disk in the configuration $s_1(C)$ can be moved at least once by the following $g(C) - 1$ $(= g(n,k) - 1)$ moves $s_2, \ldots, s_{g(C)}$, contradicting the definition of $g(n,k)$. Since disk i is moved only by $s_1 \in S_1$, we have $i \in D_1 - D_2 \neq \emptyset$. Similarly, the disk moved by the last move $s_{g(C)}$ of S cannot be moved more than once by S either. Thus $D_2 - D_1 \neq \emptyset$.

Let $D_1 - D_2 = \{r_1, \ldots, r_l\}$ and $D_2 - D_1 = \{t_1, \ldots, t_m\}$, where $1 \leq l, m \leq n-1$. We may label the disks in such a way that disk i has larger size than disk j if and only if $i > j$. Let r_1 be the smallest number in $(D_1 - D_2) \cup (D_2 - D_1)$. Since $|D_1| = |D_1 \cup D_2| - |D_2 - D_1| = n - m$, by the definition of D_1, we know that S_1 moves $n-m$ different pegs. Suppose the moves of S_1 take place in t pegs, where $t \leq k$. Then, by Lemma 1,

$$|S_1| \geq g(n-m,k). \tag{1}$$

Since $r_1 \in D_1 - D_2$, disk r_1 is not moved by S_2. Since $r_1 < t_i$ for all $1 \leq i \leq m$, all disks t_i $(1 \leq i \leq m)$ have larger sizes than disk r_1, which is idle during the whole movement of S_2. By the "smaller disk on the top of larger disk" rule, the peg occupied by disk r_1 is completely useless when each disk t_i $(1 \leq i \leq m)$ is moved by S_2. Thus the m disks $t_1, \ldots, t_m$ are moved by S_2 using at most $k-1$ pegs. (S_2 might also move disks other than disks $t_1, \ldots, t_m$. But those moves and disks can be ignored since they do not affect the moves involving disks $t_1, \ldots, t_m$. So one can focus on only the subsequence of S_2 that moves disks $t_1, \ldots, t_m$.) By Lemma 1,

$$|S_2| \geq g(m,k-1). \tag{2}$$

Note that $g(n,k) \geq 2\max\{|S_1|, |S_2|\} - 1$. If $g(n-m,k) \neq g(m,k-1)$, then Theorem 1 follows from (1) and (2). If $g(n-m,k) = g(m,k-1)$, then $g(n,k) \geq 2|S_1| \geq 2g(n-m,k) = 2\max\{g(n-m,k), g(m,k-1)\}$. $\square$

LEMMA 2. *Suppose $n \geq 1$ and $k \geq 3$. Then the function $g(n,k)$ is strictly increasing with respect to the variable n.*

Proof. Let C be an extremal configuration of n disks and k pegs with $g(C) = g(n,k)$. Let $S = \big(s_1, s_2, \ldots, s_{g(C)}\big)$ be a sequence of $g(C)$ moves that move every disk of C at least once, where all moves are taken according to the rules of the Towers of Hanoi. Let $s_1(C)$ be the configuration obtained by applying the first move s_1 of S to C. Suppose the move s_1 moves disk i. In the proof of Theorem 1, it is shown that disk i cannot be moved more than once by S. Then every disk except disk i in the configuration of $s_1(C)$ is moved at least once by $S - \{s_1\}$, which consists of a sequence of $g(n,k) - 1$ moves. Let C' be the configuration obtained by removing the disk i from the configuration $s_1(C)$. Then C' has $n-1$ disks and k pegs, and $g(n-1,k) \leq g(C') = g(C) - 1 = g(n,k) - 1$. ∎

COROLLARY 1. *Suppose $n \geq 2$ and $k \geq 4$. Then for every m with $1 \leq m \leq n-1$,*

$$g(n,k) \geq 2\min\{g(n-m,k), g(m,k-1)\}.$$

Proof. By Lemma 2, $g(n-m,k)$ is a strictly decreasing function of m, and $g(m,k-1)$ is a strictly increasing function of m. Corollary 1 obviously follows from Theorem 1. ∎

As usual, the function $\binom{x}{t}$ can be extended to real x for each integer t as follows:

$$\binom{x}{t} = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t = 0, \\ x(x-1)\cdots(x-t+1)/t! & \text{if } t > 0. \end{cases}$$

In particular, $\binom{0}{0} = 1$ by the above definition. The identity $\binom{x}{t} = \binom{x-1}{t} + \binom{x-1}{t-1}$ will be used repeatedly in the proofs.

THEOREM 2. *Suppose $k \geq 3$. Then for every integer $s \geq 2$,*

$$g\left(\binom{s}{k-2} + \binom{s+k-7}{k-5}, k\right) \geq 2^{s-2}.$$

Proof. We use double-induction on k and s. First, we use induction on k. If $k = 3$, by [9, Remark 2],

$$g\left(\binom{s}{k-2} + \binom{s+k-7}{k-5}, k\right) = g(s,3) \geq 2^{s-2} + 1$$

for all $s \geq 2$. Now suppose $k \geq 4$ and suppose the theorem is true for $k-1$; that is, for every integer $s \geq 2$,

$$g\left(\binom{s}{k-3} + \binom{s+k-8}{k-6}, k-1\right) \geq 2^{s-2}.$$

Equivalently, by using $s-1$ to replace s in the above, we have

$$g\left(\binom{s-1}{k-3} + \binom{s+k-9}{k-6}, k-1\right) \geq 2^{s-3} \tag{3}$$

for all $s \geq 3$.

Second, we use induction on s. If $s = 2$, then $\binom{s}{k-2} + \binom{s+k-7}{k-5} = \binom{2}{k-2} + \binom{k-5}{k-5} = 1$ since $k \geq 4$. Thus

$$g\left(\binom{s}{k-2} + \binom{s+k-7}{k-5}, k\right) = g(1,k) = 1 = 2^{s-2};$$

that is, Theorem 2 is true for $s = 2$ and $k \geq 4$. Now suppose $s \geq 3$ and suppose the theorem is true for $s-1$; that is, for every integer $k \geq 4$,

$$g\left(\binom{s-1}{k-2} + \binom{s+k-8}{k-5}, k\right) \geq 2^{s-3}. \tag{4}$$

Let $n = \binom{s}{k-2} + \binom{s+k-7}{k-5}$ and $m = \binom{s-1}{k-3} + \binom{s+k-8}{k-6}$. Then by Corollary 1 together with (3) and (4),

$$\begin{aligned} g(n,k) &\geq 2\min\{g(n-m,k), g(m,k-1)\} \\ &= 2\min\left\{g\left(\tbinom{s-1}{k-2} + \tbinom{s+k-8}{k-5}, k\right),\, g\left(\tbinom{s-1}{k-3} + \tbinom{s+k-8}{k-6}, k-1\right)\right\} \\ &\geq 2\min\left\{g\left(\tbinom{s-1}{k-2} + \tbinom{s+k-8}{k-5}, k\right),\, g\left(\tbinom{s-1}{k-3} + \tbinom{s+k-9}{k-6}, k-1\right)\right\} \\ &\geq 2^{s-2}; \end{aligned}$$

that is, Theorem 2 is true for s and k. The proof is complete by the principle of double-induction. $\square$

3. Proof of main result. By the definition of $g(n,k)$ and $H(n,k)$, we have $g(n,k) \leq H(n,k) \leq FS(n,k)$. Thus, in order to obtain the order of magnitude of $H(n,k)$, one needs to have a lower bound on $g(n,k)$ and an upper bound on $FS(n,k)$ with the same order of magnitude.

LEMMA 3. *Suppose $n \geq 1$ and $k \geq 3$. Then*

$$\log FS(n,k) < (n(k-2)!)^{1/(k-2)} + \log n.$$

Proof. For any fixed k and n, there is a unique s such that $\binom{k+s-3}{k-2} < n \leq \binom{k+s-2}{k-2}$. The exact expression on $FS(n,k)$ in the first line below can be found in many papers (for example, [3]).

$$\begin{aligned} FS(n,k) &= 2^s\left(n - \tbinom{k+s-3}{k-2}\right) + \sum_{t=0}^{s-1} 2^t \tbinom{k+t-3}{k-3} \\ &\leq 2^s\left(n - \tbinom{k+s-3}{k-2}\right) + \tbinom{k+s-4}{k-3} \sum_{t=0}^{s-1} 2^t \\ &< 2^s\left(n - \tbinom{k+s-3}{k-2}\right) + \tbinom{k+s-4}{k-3} 2^s \\ &= 2^s\left(n - \tbinom{k+s-3}{k-2} + \tbinom{k+s-4}{k-3}\right) \\ &= 2^s\left(n - \tbinom{k+s-4}{k-2}\right) < n2^s. \end{aligned}$$

Thus $\log FS(n,k) \leq s + \log n$. To estimate s, we have $n > \binom{k+s-3}{k-2} > s^{k-2}/(k-2)!$, which implies $s < (n(k-2)!)^{1/(k-2)}$. $\square$

LEMMA 4. *Suppose $n \geq 1$ and $k \geq 3$. Then*

$$\log g(n,k) > (n(k-2)!)^{1/(k-2)} - k + 1.$$

Proof. Lemma 4 holds for $n = 1$ since

$$\log g(1,k) = \log 1 = 0 > ((k-2)!)^{1/(k-2)} - k + 1.$$

If $k = 3$, by [9, Remark 2], we have $g(n, 3) \geq 2^{n-2} + 1$ for all $n \geq 2$. Thus

$$\log g(n, 3) > n - 2 = (n(k-2)!)^{1/(k-2)} - k + 1.$$

Now suppose $n \geq 2$ and $k \geq 4$. Then there is a unique s such that $\binom{s}{k-2} + \binom{s+k-7}{k-5} < n \leq \binom{s+1}{k-2} + \binom{s+k-6}{k-5}$. Also it is easy to verify that $s \geq 2$. To estimate s, we have

$$\begin{aligned} n &\leq \tbinom{s+1}{k-2} + \tbinom{s+k-6}{k-5} \leq \begin{cases} \tbinom{s+1}{k-2} & \text{if } k = 4, \\ \tbinom{s+k-4}{k-2} + \tbinom{s+k-4}{k-3} & \text{if } k \geq 5 \end{cases} \\ &= \tbinom{s+k-3}{k-2} < (s+k-3)^{k-2}/(k-2)!, \end{aligned}$$

from which $s > (n(k-2)!)^{1/(k-2)} - k + 3$. By Lemma 2 and Theorem 2,

$$\log g(n, k) > \log g\left(\binom{s}{k-2} + \binom{s+k-7}{k-5}, k\right) \geq s - 2 > (n(k-2)!)^{1/(k-2)} - k + 1. \qquad \square$$

Finally, we have the main theorem (Theorem 3) showing that $FS(n, k)$ and $H(n, k)$ both have the order of magnitude of $2^{(1\pm o(1))(n(k-2)!)^{1/(k-2)}}$.

THEOREM 3. *Suppose $n \geq 1$ and $k \geq 3$. Then*

$$\log FS(n, k) = \log H(n, k) + \Theta(k + \log n) = (n(k-2)!)^{1/(k-2)} + \Theta(k + \log n).$$

In other words, for each fixed k and for $n \gg k$,

$$FS(n, k) = 2^{(1\pm o(1))(n(k-2)!)^{1/(k-2)}} = H(n, k).$$

Proof. By the definition of $g(n, k)$ and $H(n, k)$, we have $g(n, k) \leq H(n, k) \leq FS(n, k)$. By Lemmas 3 and 4,

$$(n(k-2)!)^{1/(k-2)} - k + 1 < \log H(n, k) \leq \log FS(n, k) < (n(k-2)!)^{1/(k-2)} + \log n,$$

from which Theorem 3 follows. $\square$

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